Journal of Statistical Physics, Vol. 66, Nos. 3/4, 1992

On the Spectrum of the Linear Boltzmann Operator

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Received September 25, 1990; final April 16, 1991

The continuous spectrum of the linear Boltzmann operator with constant field is derived. It is found that a sufficiency relation for runaway phenomena is consistent with another sufficiency relation for the hydrodynamic regime to exist. There is a further class of systems whose behavior lies in between these two extremes.

KEY WORDS: Spectrum of linear Boltzmann operator; hydrodynamic regime; runaway phenomena.

1. INTRODUCTION

Knowledge of the structure of the spectrum of the linear Boltzmann operator for the spatially inhomogeneous and non-zero-field cases is useful in the theory of electron swarms.[1] The spectrum is only known for a few special cases of the molecular potential[2,3] and then only for the collision operator $\mathscr{J}$. Nicolaenko[4] derives the properties of the spatially inhomogeneous hard-sphere operator. We will concern ourselves with the non-zero-field operator, of most concern to swarm physics.

The linear Boltzmann operator is given by

$$\mathscr{B}\mathbf{f} = \mathbf{c}\cdot\nabla f + \mathbf{a}\cdot\partial_{\mathbf{c}} f + \mathscr{J} f \tag{1}$$

where $\mathbf{a}$ is the field term, and $\mathscr{T}$ is the collision operator.

We can split the collision operator into its direct and restitutional parts:

$$\mathscr{J} f = \nu(c) f - \mathscr{K} f = \nu(c) f - \int k(\mathbf{c}, \mathbf{c}')\, d\mathbf{c}' \tag{2}$$

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0022-4715/92/0200-1003$06.50/0

where[5]

$$\nu(c) = \frac{2\pi}{m_0} \int f_0(c_0)\, g\sigma(\theta, g) \sin\theta \, d\theta \, d\mathbf{c}_0$$

and

$$k(\mathbf{c}, \mathbf{c}') = \frac{1}{m_0} \left(\frac{m + m_0}{2m_0} \right)^4 \int \frac{\sin\theta}{\cos^4\theta} \sigma\left(\theta, \frac{m + m_0}{2m_0 \cos\theta} |\mathbf{c} - \mathbf{c}'| \right)$$
$$\times f_0 \left(\mathbf{c} + \frac{m}{m_0} (\mathbf{c} - \mathbf{c}') - \frac{m + m_0}{2m_0 \cos\theta} \hat{\mathbf{n}} \, |\mathbf{c} - \mathbf{c}'| \right) d\theta \, d\varepsilon$$

where σ is the differential collision cross section, $\mathbf{c}$ and $\mathbf{c}_0$ are the velocities of the two particles before the collision, and $\mathbf{g} = \mathbf{c} - \mathbf{c}_0$. The primed variables refer to velocities after the collision has occurred, and θ and ε are the polar coordinates of $\hat{\mathbf{n}}$, with the polar axis aligned with $\mathbf{c} - \mathbf{c}'$. The collision frequency ν is spherically symmetric, as only two-body collision are taken into account. Under quite general circumstances, $\mathscr{K}$ is compact, even though an arbitrary cutoff must often be applied to the potential in order for $\nu(c)$ and $\mathscr{K}$ to be defined. For example, if k is Lebesgue square-integrable, then K is trace-class, and hence compact. For the purposes of this paper, we assume that K is compact. We do this to use a generalized version of Weyl's theorem, which states that adding a compact Hermitian operator to a Hermitian operator does not alter the continuous part of the spectrum.[6] We can apply this to the collision operator to conclude that the continuous spectrum of $\mathscr{J}$ is just that of $\nu(c)$, i.e., the range of values that $\nu(c)$ takes. For the hard sphere, and hard-power-law potentials $[V(r) \propto r^{-s}$ with $s > 5]$, $\nu(c)$ has a minimum at $c = 0$ and increases monotonically with c. For soft power-law potentials ($s < 5$), $\nu(c)$ has a maximum at $c = 0$ and decreases monotonically toward 0. The case of Maxwell molecules ($s = 5$) is interesting because $\nu(c)$ is a constant function, and so $\mathscr{J}$ has a pure point spectrum.[2]

This paper is concerned with generalizing the above results to the spatially inhomogeneous and non-zero-field case. A lot of care must be taken in this case, as the operators are no longer self-adjoint, and the essential spectrum that remains invariant under compact perturbation and is not so easily characterized.

The final point to be made clear is the space on which the operator acts. The only property that we really need to use is that its elements $\phi(\mathbf{c})$ should remain bounded as $c \to \infty$. This obviously includes any of the spaces $L^p(\mathbb{R}^3)$, $p \in [1, \infty]$. However, to make the discussion of spectral properties simpler, we shall specialize to the Hilbert space $L^2(\mathbb{R}^3)$ even though $\hat{L}^1(\mathbb{R}^3)$ is arguably a more natural choice.

2. SPATIALLY INHOMOGENEOUS AND NON-ZERO-FIELD CASES

Discussing the spectrum of the Boltzmann operator is equivalent to discussing the spectrum of the Fourier transform in position space, i.e., it is the same as the union over **k** of the spectra of

$$\mathscr{L} = i\mathbf{c}\cdot\mathbf{k} + \mathbf{a}\cdot\partial_{\mathbf{c}} + \nu(c)$$

where $\mathbf{k} \in \mathbb{R}^3$.

It is convenient at this point to introduce a cylindrical coordinate system in which c_z is the component of **c** in the direction of **a** and $c_\perp$ is the component transverse to it, and c_θ is the angular component. Introducing the function

$$w(\mathbf{c}) = \exp\left(-\frac{\operatorname{sgn}(c_z)}{a}\int_0^c \frac{\nu(c')c'}{(c'^2 - c_\perp^2)^{1/2}}\,d\mathbf{c}' - \frac{ic_z\mathbf{c}\cdot\mathbf{k}}{a} + \frac{ic_z^2 k_z}{2a}\right) \tag{3}$$

then we can relate $\mathscr{L}$ to the derivative operator by means of

$$\mathscr{L} = w\mathbf{a}\cdot\partial_{\mathbf{c}} w^{-1} \tag{4}$$

Let λ_0 be defined by

$$\lambda_0 = \lim_{c\to\infty}\frac{1}{c}\int_0^c \nu(c')\,dc' = \lim_{c\to\infty}\nu(c) \tag{5}$$

Now the resolvant may be written explicitly:

$$(\mathscr{L}-\lambda)^{-1}\phi = \begin{cases} -e^{\lambda c_z}w(\mathbf{c})\displaystyle\int_{c_z}^{\infty}\frac{e^{-\lambda u}}{w(c_\perp, u)}\phi(u)\,du, & \operatorname{Re}\lambda > \lambda_0 \\ e^{\lambda c_z}w(\mathbf{c})\displaystyle\int_{-\infty}^{c_z}\frac{e^{-\lambda u}}{w(c_\perp, u)}\phi(u)\,du, & \operatorname{Re}\lambda < \lambda_0 \end{cases} \tag{6}$$

We shall consider the case of $\operatorname{Re}\lambda = \lambda_0$ later. We can rewrite the resolvant as an integral operator with kernel

$$r(c_z, c_z') = \begin{cases} -\delta(c_\perp - c_\perp')\,\Theta(c_z' - c_z)\dfrac{w(\mathbf{c})}{w(\mathbf{c}')}e^{\lambda(c_z - c_z')}, & \operatorname{Re}\lambda > \lambda_0 \\ \delta(c_\perp - c_\perp')\,\Theta(c_z - c_z')\dfrac{w(\mathbf{c})}{w(\mathbf{c}')}e^{\lambda(c_z - c_z')}, & \operatorname{Re}\lambda < \lambda_0 \end{cases} \tag{7}$$

In the first case, the kernel is nonzero only when $c_z' > c_z$. As $c_z \to \infty$, the term

$$e^{\lambda(c_z - c_z')}w(\mathbf{c})/w(\mathbf{c}') \sim e^{(\lambda-\lambda_0)(c_z - c_z')} \to 0$$

Since w remains bounded as $c_\perp \to \infty$, it is clear that the kernel is integrable with respect to c, and that $\int r(\mathbf{c}, \mathbf{c}')\, d\mathbf{c}$ is a bounded function. A similar argument holds in the case of $\operatorname{Re} \lambda < \lambda_0$, so the kernel r is a bounded function, which means that the resolvant operator $(\mathscr{L} - \lambda)^{-1}$ is bounded, and so the set of all λ with real part not equal to λ_0 lies within the resolvant set.

The situation when $\operatorname{Re} \lambda = \lambda_0$ depends on the asymptotic properties of

$$\gamma = \left(\lambda_0 - \frac{1}{a|c_z|} \int_0^{c_z} v(c')\, dc' \right) c_z \qquad \text{as} \quad c_z \to \infty$$

If this term diverges, then so does w or its inverse. One can choose whichever of the cases in Eq. (6) that is defined—if the resulting kernel is integrable, then the resolvant is bounded; if not, then the resolvant is unbounded, and the line $\operatorname{Re} \lambda = \lambda_0$ is the continuous part of the spectrum of $\mathscr{L}$. In the case where γ converges to a finite quantity [for example, if $v(c)$ is analytic at ∞], then $\mathscr{L}$ is clearly related by a similarity transformation to $\mathbf{a} \cdot \partial_\mathbf{c} + \lambda_0$, and so has the same spectrum (a continuous spectrum along the line $\operatorname{Re} \lambda = \lambda_0$).

In order to discuss the spectrum of $\mathscr{L} + \mathscr{K}$, we need to recall some facts from Fredholm operator theory.[7] A Fredholm operator $\mathscr{A}$ is closed operator satisfying:

1. Range($\mathscr{A}$) is closed.
2. $\alpha(\mathscr{A}) < \infty$, where α is the nullity of $\mathscr{A}$.
3. $\beta(\mathscr{A}) < \infty$, where β is the deficiency of $\mathscr{A}$.

The index of $\mathscr{A}$ is given by

$$i(\mathscr{A}) = \alpha(\mathscr{A}) - \beta(\mathscr{A})$$

For arbitrary $\mathscr{A}$ the Φ-set, denoted by $\Phi_\mathscr{A}$, is the set of those complex λ for which $\mathscr{A} - \lambda$ is a Fredholm operator. The properties that we will use are:

1. $\Phi_\mathscr{A}$ is open.
2. If $\mathscr{A}$ is closed, and $\mathscr{B}$ is $\mathscr{A}$-compact, then $\Phi_\mathscr{A} = \Phi_{\mathscr{A}+\mathscr{B}}$ and
$$i(\mathscr{A} - \lambda) = i(\mathscr{A} + \mathscr{B} - \lambda) \qquad \text{for} \quad \lambda \in \Phi_\mathscr{A}$$
3. $i(\mathscr{A} - \lambda)$ is constant on any component of $\Phi_\mathscr{A}$.
4. $\alpha(\mathscr{A} - \lambda)$ and $\beta(\mathscr{A} - \lambda)$ are constant on any component of $\Phi_\mathscr{A}$ except possibly on a discrete set of points at which they have larger values.

The essential spectrum according to Wolf,[8] σ_{ew}, is defined to be the complement of $\Phi_{\mathscr{A}}$ in the complex plane. In the case of $\mathscr{L}$, the essential spectrum $\sigma_{ew}(\mathscr{L})$ is the line given by $\operatorname{Re}\lambda = \lambda_0$, unless

$$\int_{-\infty}^{0} w(c) e^{\lambda_0 c_z} \, dc_z$$

converges, in which case, $\sigma_{ew}(\mathscr{L}) = \varnothing$.

Since $\mathscr{K}$ is compact, $\Phi_{\mathscr{L}-\mathscr{K}} = \Phi_{\mathscr{L}}$ consists of the two half-planes separated by the line $\operatorname{Re}\lambda = \lambda_0$. We also know that the index $i(\mathscr{L} - \mathscr{K} - \lambda) = 0$, $\forall \lambda \in \Phi_{\mathscr{L}}$. It remains to be shown that $\alpha(\mathscr{L} - \mathscr{K} - \lambda) = 0$ (or equivalently β) on any open subset of $\Phi_{\mathscr{L}}$. Then there remains only a discrete set of points at which the nullity may differ from zero—these correspond to the discrete eigenvalues of $\mathscr{L} - \mathscr{K}$.

Since the two sections of $\Phi_{\mathscr{L}}$ are half-planes, we can choose $\lambda \in \Phi_{\mathscr{L}}$ with $|\lambda|$ sufficiently large to ensure that $\|(\mathscr{L} - \lambda)^{-1}\| < \|\mathscr{K}\|^{-1}$. Then we can write

$$(\mathscr{L} - \lambda - \mathscr{K})^{-1} = (\mathscr{L} - \lambda)^{-1} \sum_{j=0}^{\infty} (\mathscr{L} - \lambda)^{-j} \mathscr{K}^{j} \tag{8}$$

which converges to a bounded operator. This means that there is in both parts of $\Phi_{\mathscr{L}}$ an open subset that is part of the resolvant of $\mathscr{L} - \mathscr{K}$, and hence $\alpha(\mathscr{L} - \mathscr{K} - \lambda) = 0$ on this open set. To summarize this section, we have proved that if $\mathscr{K}$ is compact, $\mathscr{B} = \mathscr{L} - \mathscr{K}$ has the same continuous spectrum as $\mathscr{L}$, namely the line $\operatorname{Re}\lambda = \lambda_0$, unless

$$\int_{-\infty}^{0} w(\mathbf{c}) e^{\lambda_0 c_z} \, dc_z < \infty$$

in which case there is no continuous spectrum.

3. HYDRODYNAMICS AND RUNAWAYS

In the previous section, the continuous spectrum of the Boltzmann operator was linked to the velocity-dependent collision frequency. This is important in the theory of swarms, as it is known that if the spectrum has an isolated eigenvalue at zero, then a hydrodynamic regime exists where the transport coefficients such as drift velocity approach a constant value over a characteristic time.[9] Conversely, it is necessary for the spectrum to be continuous at zero for runaway to occur, where the swarm particles are accelerated forever, and no meaning can be attached to drift velocity.

On the other hand, Cavalleri and Paveri-Fontana[10,11] have shown that the existence of $\int_0^\infty \nu(c')\,dc'$ is sufficient for the runaway effect to occur. From Eq. (5) we see that the condition of the spectrum being continuous at zero is equivalent to saying that $\lim_{c\to\infty} c^{-1}\int_0^c \nu(c')\,dc' = 0$, and this is not the same as Cavalleri and Paveri-Fontana's condition. There is a class of system for which $\int_0^c \nu(c')\,dc'$ diverges sublinearly that is not clearly hydrodynamic or runaway.

We can study this phenomenon with a one-dimensional exactly solvable model, which is a generalization of the classical BGK model. In this model we take the BGK form of the restitutional part of the collision operator, i.e., a projection onto the background equilibrium distribution:

$$\mathscr{K}f = \nu(c)\, f_0(c) \int f(c, t)\, dc \tag{9}$$

We can simplify the solution of this model with the spatial average $f^{(0)}(c) = \int f(c, r)\, dr$, as it contains sufficient information to calculate the transport coefficients, for example, drift velocity, by $v_{\mathrm{dr}} = \int c f^{(0)}(c)\, dc$.[1]

The model equation now reads [dropping the superscript (0)]

$$[\partial_t + a\partial_c + \nu(c)]\, f(c, t) - \nu(c)\, f_0(c) = 0 \tag{10}$$

If we make the substitution

$$f(c, t) = \exp\left(-\frac{1}{a}\int_0^c \nu(c')\, dc'\right) f'(c, t) \tag{11}$$

then Eq. (10) is transformed into

$$(\partial_t + a\partial_c)\, f'(c, t) = \nu(c)\, f_0(c) \exp\left(\frac{1}{a}\int_0^c \nu(c')\, dc'\right) \tag{12}$$

The homogeneous part of this equation is clearly satisfied by a simularity solution, and a particular solution is given by a time-independent solution, so the general solution reads

$$f'(c, t) = \xi(c - at) + \frac{1}{a}\int_0^c \nu(c')\, f_0(c') \exp\left(\frac{1}{a}\int_0^{c'} \nu(c'')\, dc''\right) dc' \tag{13}$$

where $\xi(c)$ is defined by the value of f at time $t = 0$:

$$\xi(c) = \exp\left(\frac{1}{a}\int_0^c \nu(c')\, dc'\right) f(c, 0)$$
$$- \frac{1}{a}\int_0^c \nu(c')\, f_0(c') \exp\left(\frac{1}{a}\int_0^{c'} \nu(c'')\, dc''\right) dc'$$

When this is substituted into (11), we find

$$f c, t) = \exp\left(-\frac{1}{a}\int_0^{at-c} \nu(c')\,dc'\right) f(c-at, 0) + \exp\left(-\frac{1}{a}\int_0^{c} \nu(c')\,dc'\right)\frac{1}{a}\int_{c-at}^{c} \nu(c') f_0(c') \times \exp\left(\frac{1}{a}\int_0^{c'} \nu(c'')\,dc''\right) dc' \tag{14}$$

The first term of the solution depends on the initial valocity distribution, and so is "nonhydrodynamic." The second term is independent of the initial conditions, and as $t \to \infty$, the integrand is cut off by $f_0(c') \exp[(1/a) \int_0^{c'} \nu(c'')\,dc'']$ as $c' \to -\infty$, and so converges to a time-independent solution. We can identify this term with the hydrodynamic solution.

Let us now look at the evolution of the drift velocity from a delta initial distribution $f(c, 0) = \delta(c - c_0)$:

$$v_{\text{dr}} = \int cf(\mathbf{c}) = (at + c_0) \exp\left(-\frac{1}{a}\int_{-c_0}^{at+c_0} \nu(c')\,dc'\right) + v_1(t) \tag{15}$$

where $v_1(t)$ converges to a finite value as $t \to \infty$.

Cavalleri and Paveri-Fontana's condition (that the integral exists as $t \to \infty$) corresponds to a "ballistic" regime, in which the swarm is accelerating with acceleration $a \exp[-(1/a) \int_{-c_0}^{\infty} \nu(c')\,dc']$. If the integral diverges slower than the natural logarithm, then this term dominates in the infinite-time limit, and the drift velocity diverges sublinearly. However, if the integral diverges faster than the natural logarithm, the drift velocity converges to a finite value at $t = \infty$, but the convergence is over an infinite time scale (i.e., slower than exponential) unless the divergence of the integral is at least linear.

Another point to be made is that even though this theory encompasses the conditions by Cavalleri and Paveri-Fontana, and also the Waldman–Mason[12] theory, a major drawback is immediately visible upon comparison with the experimental data.[13–15] This is that λ_0 is independent of the ratio E/N, as is Cavalleri–Paveri-Fontana condition; however, the experimental data show a marked contrast between regions of E/N where the behavior of the system is hydrodynamic and those where runaway sets in. The only situation where this sort of behavior can be allowed theoretically is when $\lambda_0 = 0$ and the Cavalleri-Paveri–Fontana condition does not hold. This is certainly the case with the experiments of Ness and

Robson[15] in water vapor. In this case, $\nu(c) \sim c^{-1}$. A more sophisticated theory is clearly needed in this case, whereby when E/N is small, an approximate Boltzmann equation is used that reflects the dominance of a rising collision frequency; at intermediate E/N, the approximation reflects the sampling of the fast decay in $\nu(c)$; and finally at very high E/N the approximation will reflect the quenching of the runaway by inelastic processes.

4. CONCLUSION

In this paper, the continuous spectrum of the linear Boltzmann operator with a field or spatial gradient term has been related to the velocity-dependent collision frequency. This has demonstrated the consistency of Cavalleri and Paveri-Fontana's[10] result with that of Standish.[9] It has put runaway phenomena and hydrodynamic phenomena as two ends of a continuum, with a class of nonhydrodynamic phenomena in between.

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